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Stability analysis for the quartic Landau–Ginzburg model

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Abstract. The three-dimensional stability of static one-dimensional solutions of the Landau–Ginzburg equation is investigated. A general formula for the growth or decay rate is obtained for arbitrary local free energies, using a method developed earlier by Rowlands and Infeld for different equations. The results are then applied to quartic free energies. Among the real finite solutions obtained earlier, cnoidal waves are shown to be stable, and solitary waves and kinks marginally stable. Two other types of non-linear periodic waves are shown to be unstable.

1. Introduction

Some recent papers have been devoted to the form of the possible spatial structure of magnetic phase transitions (Khan 1986, Winternitz *et al* 1988, Skierski *et al* 1988). These workers used the Landau–Ginzburg equation to describe the order parameter in the magnetic material. This equation may be written in the form

$$(1/\Gamma)(\partial M/\partial t) = 2\,\nabla^2 M - \dot{V}(M) \qquad \Gamma > 0 \tag{1.1}$$

where M is the magnetisation, Γ is the Landau–Khalatnikov damping coefficient, which sets the scale of the relaxation process, and the dot denotes a derivative with respect to the argument. The term on the left-hand side describes the relaxation of the system, while $V(M)$ is the local free energy (see, e.g., Landau and Lifshitz 1980, Cyrot 1973).

The above-cited workers have studied a wide class of exact analytic solutions of equation (1.1), and in particular time-independent ones.

The situations studied include biquadratic functions $V(M)$. Among the obtained solutions are spatially one-dimensional ones, and in particular several different types of periodic non-linear waves (expressed in terms of the Jacobi elliptic functions $\mathrm{dn}(x, k)$, $\mathrm{sn}(x, k)$ and $\mathrm{cn}(x, k)$) and also solitary pulses (domains) and kinks (Winternitz *et al* 1988). The method used was that of symmetry reduction, coupled with singularity analysis. The same method has been applied to obtain exact solutions of a variety of different non-linear Klein–Gordon and Schrödinger equations (Winternitz *et al* 1987, Gagnon and Winternitz 1989a, b, c).

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The purpose of this paper is to study the stability and the relaxation of the obtained structures. More specifically, we study the stability of one-dimensional static solutions $M = M_0(x)$ of the Landau–Ginzburg equation (1.1), which for $M_0(x)$ reduces to an ordinary differential equation. Integrating it once, we obtain the equation

$$(M_{0,x})^2 - V(M_0) = a \qquad (1.2)$$

where a is a real integration constant, related to the amplitude of the spatial variations of M_0. The quantity a is also directly related to the free energy at the point x_0 for which we have $M_{0x}(x_0) = 0$. Letter subscripts in (1.2) and below denote derivatives.

The study of the stability of other than one-dimensional solutions is of considerable interest but goes beyond the scope of this article.

A constant solution of equation (1.1) (or (1.2)) is defined by $V(M_0) = 0$. The relaxation to such a spatially homogeneous and time-independent equilibrium is readily obtained by linearising about this state and solving the linearised equation to give

$$M(x, t) = M_0 + \psi_0 \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) \exp\{-\Gamma[2|\boldsymbol{k}|^2 + \ddot{V}(M_0)]t\} \qquad \psi_0 = \text{constant}. \qquad (1.3)$$

For a constant equilibrium state where the free energy is a minimum $(\ddot{V}(M_0) > 0)$ we see that any perturbation of the considered type relaxes back to the equilibrium state and that the presence of a spatial structure in the perturbation $(|\boldsymbol{k}|^2 > 0)$ enhances the relaxation rate. For an energy maximum $\ddot{V}(M_0) < 0$, the system will only relax to equilibrium for sufficiently short-wavelength disturbances $(|\boldsymbol{k}|^2 > |\ddot{V}|/2)$. Otherwise the system is unstable and small disturbances increase exponentially in time.

In order to study the stability and relaxation rates for time-independent but spatially varying solutions of (1.2), we apply a method of analysis originally introduced to study stability problems in plasma physics by Infeld and Rowlands (1979) (for a review see Infeld and Rowlands (1990)). A large body of work exists on the stability of solutions of non-linear equations. As somewhat similar in spirit, we mention the work of Harrowell and Oxtoby (1987) on perturbations of solitons. For soliton stability, see also Zakharov *et al* (1986).

The method is relatively straightforward and so the present paper is self-contained. In section 2 we perform the analysis for arbitrary forms of $V(M)$, when M_0 is a periodic solution of equation (1.2). The case when $V(M)$ is biquadratic is studied in section 3.

The result of the analysis is that the linearisation of equation (1.1) about non-constant spatially periodic static equilibria leads to perturbations of the form

$$\delta M(\boldsymbol{x}, t) = \psi(x) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \gamma t) \qquad (1.4)$$

where $\psi(x)$ is a periodic function of x with the same period as $M_0(x)$, $\boldsymbol{k}$ is an arbitrarily oriented small wavevector and γ is a small growth (or decay) rate.

We obtain

$$\gamma/\Gamma = -(\lambda^2/\lambda_a G)k_\parallel^2 + 2k_\perp^2 \qquad k_\parallel = k_x \quad k_\parallel^2 + k_\perp^2 = \boldsymbol{k}^2 \qquad (1.5)$$

where λ is the wavelength of the non-linear equilibrium (solitons and kinks are considered as $\lambda \to \infty$ limits in this treatment), $\lambda_a \equiv \mathrm{d}\lambda/\mathrm{d}a$ and G is a positive definite quantity.

The result (1.5) is similar to one obtained by Infeld and Rowlands (1979) in the context of a non-linear Klein–Gordon equation. The basic implication of the result (1.5) is the following.

(i) For $\mathrm{d}\lambda/\mathrm{d}a < 0$, waves are always stable.

(ii) For $\mathrm{d}\lambda/\mathrm{d}a > 0$, there is a range of angles θ between the wavevector $\boldsymbol{k}$ of the perturbation and the x axis (along which we have the basic structure $M_0(x)$) yielding instability, namely $M_0(x)$ is then instable for

$$(k_\perp^2/k_\parallel) < \lambda^2/2\lambda_a G. \qquad (1.6)$$

2. Basic formula for the growth rate

Let us denote a static periodic solution of the reduced equation (1.2) as $M_0(x, a)$, indicating its dependence on the integration constant a. To study the relaxation of such solutions we consider small disturbances about the equilibrium state and write

$$M(\boldsymbol{x}, t, a) = M_0(x, a) + \delta M(\boldsymbol{x}, t, a). \tag{2.1}$$

We substitute (2.1) into (1.1), expand $\dot{V}(M)$ into a Taylor series and linearise in δM (assuming that $\delta M \ll M_0$), to obtain

$$-(1/\Gamma)(\delta M)_t + 2\nabla^2(\delta M) - \ddot{V}(M_0)\, \delta M = 0. \tag{2.2}$$

Since the coefficients in (2.2) are independent of time and periodic in x, we use Floquet's theorem (Ince 1956) to write

$$\delta M(\boldsymbol{x}, t, a) = \psi(x, a) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} - \gamma t) \tag{2.3}$$

where $\psi(x, a)$ is a periodic function of the same period as $M_0(x, a)$ and the wavenumber k is real, to assure that δM is bounded in space. The direction of $\boldsymbol{k}$ is arbitrary and we introduce the notation

$$k_x = k_\| = k \cos \theta \qquad k_\perp = k \sin \theta \tag{2.4}$$

where k and θ are constant.

Stability depends on the sign of γ: for $\gamma > 0$ we say that the solution $M_0(x, a)$ is stable, for $\gamma < 0$ unstable, and for $\gamma = 0$ marginally stable.

The function $\psi(x, a)$ satisfies

$$L\psi = -(\gamma/\Gamma)\psi - 4\mathrm{i}k_\|\psi_x + 2k^2\psi \qquad L = 2(\partial^2/\partial x^2) - \ddot{V}(M_0). \tag{2.5}$$

The operator L is self-adjoint over periodic functions; $\psi(x, a)$ is periodic in x with period $\lambda = \lambda(a)$.

It is usually not possible to solve eigenvalue problems such as (2.5) analytically; so we introduce a perturbative method, based on the assumption that k is small (long wavelength perturbations at arbitrary angles). We expand

$$\begin{aligned} \psi &= \varepsilon(\psi_0 + k\psi_1 + k^2\psi_2 + \dots \\ \gamma &= k\gamma_1 + k^2\gamma_2 + \dots \end{aligned} \tag{2.6}$$

and, in view of the linearisation performed, assume that $\varepsilon \ll k^2 \ll 1$. In equation (2.5) we keep terms of order 1, ε, εk, εk^2 (and drop ε^2 and εk^3). At lowest order we have $L\psi_0 = 0$ and obtain a particular solution that is periodic in x, namely

$$\psi_0 = (\partial/\partial x)[M_0(x, a)]. \tag{2.7}$$

A further solution of $L\psi_0 = 0$ that we shall need below is

$$\tilde{\psi}_0 = (\partial/\partial a)[M_0(x, a)]. \tag{2.8}$$

This solution is neither periodic nor spatially bounded. To see this, we redefine the static solution as

$$M_0(x, a) = \tilde{M}_0(\tilde{x}, \tilde{a}) \qquad \tilde{x} = x/\lambda(a) \quad \tilde{a} = a \tag{2.9}$$

where the period $\lambda(a)$ in general depends on the integration constant a (the special case when we have $\mathrm{d}\lambda/\mathrm{d}a = 0$ is analysed by Infeld and Rowlands (1979) and is of no interest in the present context). We have

$$\partial M_0(x, a)/\partial a = -[\partial \tilde{M}_0(\tilde{x}, \tilde{a})/\partial \tilde{x}](\lambda_a/\lambda^2)x + \partial \tilde{M}_0(\tilde{x}, \tilde{a})/\partial \tilde{a} \qquad \lambda_a = \mathrm{d}\lambda/\mathrm{d}a. \tag{2.10}$$

Since $\tilde{M}_0(\tilde{x}, \tilde{a})$ is periodic in $\tilde{x}$, the first term on the right-hand side is not bounded for $x \to \infty$. Thus, (2.7) is the only spatially bounded solution of $L\psi = 0$.

Proceeding to the next order in the expansion, namely εk, we obtain

$$L\psi_1 = -(\gamma_1/\Gamma)\psi_0 - 4\mathrm{i}\cos\theta\,\psi_{0x}. \tag{2.11}$$

We multiply (2.11) by ψ_0 and integrate over a period. In view of the self-adjointness of L we have $\langle\psi_0|L\psi_1\rangle = \langle\psi_1|L\psi_0\rangle = 0$, where

$$\langle\psi, \chi\rangle = \frac{1}{\lambda}\int_{x_0}^{x_0+\lambda} \psi\chi\,\mathrm{d}x. \tag{2.12}$$

Moreover, we have $\langle\psi_0, \psi_{0x}\rangle = 0$ because of periodicity and we obtain, from (2.11),

$$\gamma_1 = 0. \tag{2.13}$$

A particular solution of (2.11) with $\gamma_1 = 0$ is

$$\tilde{\psi}_1 = -\mathrm{i}x(\partial M_0/\partial x)\cos\theta. \tag{2.14}$$

This solution is, however, not periodic, nor is it bounded; it represents a secular term. In order to obtain a bounded solution of the inhomogeneous equation (2.11), we add a particular solution of the homogeneous equation $L\psi = 0$, namely one proportional to (2.10). The required solution is

$$\psi_1 = \tilde{\psi}_1 - \mathrm{i}\cos\theta(\lambda/\lambda_a)(\partial M_0/\partial a) = -\mathrm{i}(\lambda/\lambda_a)\cos\theta[\partial\tilde{M}_0(\tilde{x}, \tilde{a})/\partial\tilde{a}]. \tag{2.15}$$

The period of $\tilde{M}_0(\tilde{x}, \tilde{a})$ in $\tilde{x}$ is unity; the period of ψ_1 in x is hence λ.

The next order in the expansion is εk^2 and yields the following equation for ψ_2:

$$\begin{aligned} L\psi_2 = {} & (-\gamma_2/\Gamma + 2)(\partial M_0/\partial x) - 4\cos^2\theta(\lambda/\lambda_a)(\partial/\partial x) \\ & \times [\partial M_0/\partial a + (\partial M_0/\partial x)(\lambda_a/\lambda)x]. \end{aligned} \tag{2.16}$$

We multiply both sides of (2.16) by M_{0x}, integrate over a period λ and start the integration in (2.12) at a point x_0 satisfying

$$(\partial M_0/\partial x)|_{x=x_0} = (\partial M_0/\partial x)|_{x=x_0+\lambda} = 0. \tag{2.17}$$

Using the self-adjointness of L, we obtain

$$\left(-\frac{\gamma_2}{\Gamma} + 2\sin^2\theta\right)\oint (M_{0x})^2\,\mathrm{d}x = 4\frac{\lambda}{\lambda_a}\cos^2\theta\oint M_{0x}\left(\frac{\partial^2}{\partial x\,\partial a}M_0\right)\mathrm{d}x. \tag{2.18}$$

We now define

$$G = \oint (M_{0x})^2\,\mathrm{d}x = 2\int_{M_1}^{M_2} M_{0x}\,\mathrm{d}M_0 \tag{2.19}$$

where M_1 and M_2 are two consecutive zeros of M_{0x}, such that the integrand is non-negative for $M_1 \leqslant M \leqslant M_2$. Finally we obtain

$$\gamma_2 = -(\lambda^2\Gamma/\lambda_a G)\cos^2\theta + 2\Gamma\sin^2\theta \tag{2.20}$$

$$G = 2\int_{M_1}^{M_2}\sqrt{a + V(M_0)}\,\mathrm{d}M_0 \qquad \lambda = 2\int_{M_1}^{M_2}\frac{1}{\sqrt{a+V(M_0)}}\,\mathrm{d}M_0. \tag{2.21}$$

Returning to the expression (2.6) for the growth rate γ, we obtain equation (1.5), announced in section 1. The positive definite quantity G and the wavelength λ are given in (2.21).

The calculation of the growth rate has thus been reduced to simple quadratures for an arbitrary form of the free energy $V(M)$. As stated in section 1, the stability of the state $M_0(x, a)$ is determined by the sign of $\lambda_a = \mathrm{d}\lambda/\mathrm{d}a$.

3. Stability and growth rates for a quartic free energy

Let us now consider the case when the local free energy is a biquadratic polynomial $V(M) = -2bM^2 + cM^4$ with constant coefficients and $c \neq 0$. Exact solutions of the equations of motion were obtained by Winternitz *et al* (1988), using the method of symmetry reduction. Here we restrict ourselves to the case of finite (non-singular) real translationally invariant solutions $M(x)$. We use equation (1.5) to calculate the growth rate γ for the individual solutions and thus to establish their stability properties. For simplicity we put $k_\perp = 0$, since the actual quantity to be calculated is $\lambda^2/\lambda_a G$ and the expression for γ with $k_\perp \neq 0$ is then directly recovered from (1.5).

Equation (1.2) for the exact solution can in this case be written as

$$\dot{M}^2 = a + 2bM^2 - cM^4 \tag{3.1}$$

(we drop the subscript on M_0). Equation (3.1) is represented by the phase diagrams in figure 1 and a is the value of the intercept of the curve with the $M = 0$ axis.

Equation (3.1) can be rewritten as

$$\dot{M}^2 = -c(M^2 - M_1^2)(M^2 - M_2^2) \equiv P(M) \qquad M_{1,2}^2 = (b \mp \sqrt{b^2 + ac})/c. \tag{3.2}$$

Let us now consider the individual solutions, corresponding to the different parts of figure 1.

3.1. *Four real roots of $P(M)$ and $c > 0$*

We must have $a < 0$, $b > 0$ and have chosen $0 < M_1 < M_2$ (figure 1(a)). We have

$$\lambda = \frac{2}{\sqrt{c}} \int_{M_1}^{M_2} \frac{\mathrm{d}M}{[(M^2 - M_1^2)(M_2^2 - M^2)]^{1/2}} = \frac{\sqrt{2}}{\sqrt{b}} \sqrt{2 - q^2} K(q) \tag{3.3}$$

$$q^2 = 1 - M_1^2/M_2^2 \qquad 0 < q^2 < 1 \tag{3.4}$$

where $K(q)$ is the complete elliptic integral of the first kind (Byrd and Friedman 1971). Using (2.21) we find the expression for G in terms of first- and second-kind complete elliptic integrals to be

$$G = (4\sqrt{2}/3c)[b/(2 - q^2)]^{3/2}[(2 - q^2)E(q) - 2(1 - q^2)K(q)]. \tag{3.5}$$

Calculating the derivative of the wavelength, we find that

$$\begin{aligned}(\mathrm{d}\lambda/\mathrm{d}a) = (\mathrm{d}\lambda/\mathrm{d}q)(\mathrm{d}q/\mathrm{d}a) &= (c/4\sqrt{2}b^{5/2})[(2 - q^2)^{5/2}/q^4(1 - q^2)] \\ &\times [(2 - q^2)E(q) - 2(1 - q^2)K(q)].\end{aligned} \tag{3.6}$$

The final result for $k_\perp = 0$ is

$$\gamma/\Gamma k^2 = -6[K(q)]^2 q^4(1 - q^2)/[(2 - q^2)E(q) - 2(1 - q^2)K(q)]^2. \tag{3.7}$$

In view of (3.7), γ is negative definite and hence the corresponding periodic solution

$$M = \varepsilon M_2 \, \mathrm{dn}(\sqrt{c}M_2(x - x_0), q) \qquad \varepsilon = \pm 1 \tag{3.8}$$

is unstable.

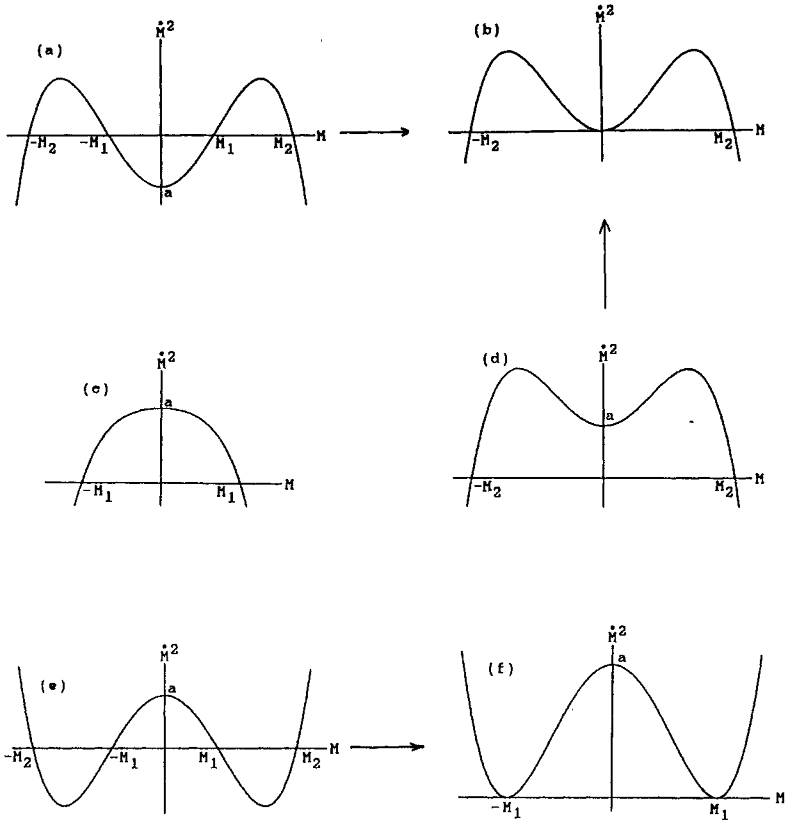

Figure 1. Phase diagrams $\dot{M}^2$ versus M corresponding to real finite solutions of equation (3.1): (*a*) $a<0, b>0, c>0$; (*b*) $a=0, b>0, c>0$; (*c*) $a>0, b<0, c>0$; (*d*) $a>0, b>0, c>0$; (*e*) $a>0, b<0, c<0, b^2>a|c|$; (*f*) $a>0, b<0, c<0, b^2=a|c|$. The arrows between the figures indicate the limits of solutions for $\lambda\to\infty$, where λ is the wavelength for periodic solutions.

The soliton limit of the solution (3.8) is obtained by taking $a\to 0$, i.e. $q^2\to 1$ (figure 1(*b*)). In this case we obtain $\gamma\to 0$ so that the soliton solution

$$M = \varepsilon M_2/\cosh[\sqrt{c}M_2(x-x_0)]. \tag{3.9}$$

is marginally stable.

3.2. *Two real and two imaginary roots of P(M) and c > 0*

In this case we have $a>0$, and the phase diagrams correspond to figure 1(*c*) or figure 1(*d*), depending on whether we have $b<0$ or $b>0$. A similar calculation as in the previous case provides, for $k_\perp=0$, the following growth rate and wavelength:

$$\begin{aligned}
&\gamma/\Gamma k^2 = 3\{r^2(1-r^2)/[(1-2r^2)E(r)/K(r)-(1-r^2)]^2\}\\
&r^2 = M_1^2/(M_1^2+|M_2|^2) \qquad M_1>0 \qquad M_2 = \mathrm{i}|M_2| \qquad\qquad (3.10)\\
&\lambda = 4[(2r^2-1)/b]^{1/2}K(r).
\end{aligned}$$

For $b<0$ we have $0<r^2<\frac{1}{2}$; for $b>0$ we have $\frac{1}{2}<r^2<1$. In both cases we obtain $\gamma>0$ and the considered periodic solutions

$$M(x) = M_1 \operatorname{cn}[\sqrt{c(M_1^2+|M_2|^2)}(x-x_0), r] \tag{3.11}$$

The soliton limit is obtained by taking $|M_2|\to 0$ (figure 1(d) becomes figure 1(b)), i.e. $r\to 1$ and $\gamma\to 0$. We again see that the solution (3.9) is marginally stable.

3.3. Four real roots of $P(M)$ and $c<0$

In (3.1) we have $c=-|c|<0$, $a>0$, $b<0$ and the corresponding phase diagram is figure 1(e). A real finite periodic solution satisfies $-M_1 \leqslant M \leqslant M_1 < M_2$ and is given by

$$M = M_1 \operatorname{sn}[\sqrt{|c|}M_2(x-x_0), q] \qquad q = M_1/M_2 \quad 0<q<1. \tag{3.12}$$

The wavelength and growth rate for $k_\perp = 0$ calculated according to (2.20) and (2.21) are

$$\lambda = (2\sqrt{2}/\sqrt{-b})\sqrt{1+q^2}K(q) \tag{3.13}$$

$$\gamma/\Gamma k^2 = -6\{q^2(1-q^2)/[(q^2+1)E(q)/K(q)-(1-q^2)]^2\} \tag{3.14}$$

and we see that the corresponding solution is unstable.

The kink solution corresponds to the limit $q\to 1$, i.e. to figure 1(f). Taking the limit in (3.13) and (3.14), we obtain $\lambda\to\infty$, $\gamma\to 0$; so the kink

$$M = \pm M_1 \tanh[\sqrt{|c|}M_1(x-x_0)] \tag{3.15}$$

is marginally stable.

4. Conclusions

A systematic application of group theory makes it possible to obtain large numbers of particular solutions of many non-linear non-integrable equations. Whether these solutions, corresponding to specific initial and boundary conditions, are actually observable in nature depends to a large degree on their stability. Furthermore, stable solutions provide a good basis for further perturbation theory calculations. These should in turn provide good approximate solutions relevant for situations in which the group theoretical solutions no longer apply. Expressions of the type (1.5) for the growth rate should play an important role in this context.

Turning to the results in section 3 for equation (3.1), we see that, of the three types of finite periodic solution obtained by Winternitz *et al* (1988), only one type is stable. These are the 'cnoidal' waves (3.11) corresponding to figures 1(c) and 1(d), and the corresponding antiferromagnetic spin waves should be observable. The solitary waves corresponding to figure 1(b) (nucleation centres of magnetic order) and the kinks in figure 1(f) (Bloch domain walls) are marginally stable and should thus also be observable.

An intuitive understanding of the stability situation can be obtained directly from the phase diagrams in figure 1. A stable solution is obtained if we have $\mathrm{d}\lambda/\mathrm{d}a<0$, i.e. if the wavelength λ decreases as the intercept a increases. If the curve in figure 1(a) is raised, a increases (towards $a=0$, since it is negative) and figure 1(a) turns into figure 1(b): a solitary wave with $\lambda\to\infty$. Thus, for figure 1(a) we have $\mathrm{d}\lambda/\mathrm{d}a>0$ and the solution is unstable. Similarly, if the curve in figure 1(e) is raised, a increases and the solution approaches the kink ($\lambda \to \infty$) in figure 1(f). Again we have $\mathrm{d}\lambda/\mathrm{d}a > 0$ and an unstable solution. On the other hand, if the curve in figure 1(d) is lowered, a decreases and we have $\lambda \to \infty$ as the solution approaches the solitary waves in figure 1(b). The corresponding solutions are stable.

In a future publication we plan to study the stability of solutions in the case when the right-hand side of (3.1) is a sixth-order polynomial. We also plan to extend the general results to the case of complex order parameters (e.g. for non-linear Schrödinger equations), and to the case of other types of solution of equation (1.1), and in particular rotationally or cylindrically symmetrical ones.

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